§ 3.3 Cramer's Rule, Volume, and Linear Transformations

As it turns out we can use determinants to solve matrix equations $Ax=b$ if A is $n \times n$. If we let $a_1, \dots, a_n$ denote the columns of A, let $\underline{A_i(b)}$ denote the matrix obtained by replacing column i of A with b

$$A_i(b) = \left[a_1 \,\middle|\, \cdots \,\middle|\, a_{i-1} \,\middle|\, b \,\middle|\, a_{i+1} \,\middle|\, \cdots \,\middle|\, a_n \right]$$

notice this is still an $n \times n$ matrix.

Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any b in $\mathbb{R}^n$, the unique solution to $Ax=b$ is

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{where} \quad x_i = \frac{\det(A_i(b))}{\det(A)} \qquad i = 1, 2, \dots, n$$

Pros

- Don't have to row reduce to solve for x

Cons

- Need to compute $n+1$ determinants

Example

Solve the system $\begin{cases} 4x_1 + x_2 = 6 \\ 3x_1 + 2x_2 = 7 \end{cases}$

$$\underbrace{\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 \\ 7 \end{bmatrix}}_{b}$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\det \begin{bmatrix} 6 & 1 \\ 7 & 2 \end{bmatrix}}{\det \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{12-7}{8-3} = \frac{5}{5} = \boxed{1}$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{\det \begin{bmatrix} 4 & 6 \\ 3 & 7 \end{bmatrix}}{5} = \frac{28-18}{5} = \frac{10}{5} = \boxed{2}$$

already computed det A

thus $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example

Find the values of parameter t for which the system

$$\begin{cases} 12x_1 + 5t\,x_2 = 2 \\ 3t\,x_1 + 5x_2 = 3 \end{cases}$$

and describe the solutions.

(Problems like this arise in electrical engineering according to the textbook...)

system has unique solution when $\det \begin{bmatrix} 12 & 5t \\ 3t & 5 \end{bmatrix} \neq 0$

i.e. when $60 - 15t^2 \neq 0$

$15(2-t)(2+t) \neq 0$ so when $\boxed{t \neq \pm 2}$

In this case since $\begin{bmatrix} 12 & 5t \\ 3t & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$x_1 = \frac{\det \begin{bmatrix} 2 & 5t \\ 3 & 5 \end{bmatrix}}{\det \begin{bmatrix} 12 & 5t \\ 3t & 5 \end{bmatrix}} = \frac{10 - 15t}{60 - 15t^2}$$

whenever $t \neq \pm 2$

$$x_2 = \frac{\det \begin{bmatrix} 12 & 2 \\ 3t & 3 \end{bmatrix}}{\det \begin{bmatrix} 12 & 5t \\ 3t & 5 \end{bmatrix}} = \frac{36 - 6t}{60 - 15t^2}$$

Recall if A is $n \times n$ and $\det A \neq 0$, then A is invertible, i.e. A^{-1} exists. However this data doesn't tell you what A^{-1} is on its own.

Now that we have Cramer's ~~Rule~~ Rule, we can produce a formula for A^{-1} in terms of $\det A$ and cofactors!

Let A be an invertible $n \times n$ matrix and let $e_1, \dots, e_n$ denote the columns of I_n. Since $AA^{-1} = I_n = [e_1 \ \cdots \ e_n]$, the j^{th} column of A^{-1} is a vector x such that

$$Ax = e_j$$

(remember we defined matrix products column by column!)

By Cramer's rule, the i^{th} entry (of the j^{th} column of A^{-1}) is

$$x_i = \frac{\det A_i(e_j)}{\det A} = \frac{(-1)^{i+j} \det A_{ji}}{\det A} = \frac{C_{ji}}{\det A}$$

The fact that $\det A_i(e_j) = (-1)^{i+j} \det A_{ji}$ can be seen easily by a cofactor expansion along e_j

$$\det A_i(e_j) = \det \begin{array}{cc} & \left[\begin{array}{ccccc} | & | & 0 & | & | \\ a_1 & \cdots & \vdots & \cdots & a_n \\ \text{row } j \ \ \sim\sim & \sim\sim & 1 & \sim\sim & \sim\sim \\ | & | & \vdots & | & | \\ | & | & 0 & | & | \end{array}\right] \\ & \text{column } i \end{array} = (-1)^{i+j} \det A_{ji}$$

Thus we have a formula for the entries of A^{-1}.

If $A^{-1} = (b_{ij})$, then $\underset{\uparrow \, (i,j) \text{ entry}}{b_{ij}} = \dfrac{C_{ji}}{\det A}$ $\leftarrow$ (j,i) cofactor of A

So $$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}$$

This matrix of cofactors is called the <u>adjugate</u> of A, $\operatorname{adj}(A)$. Thus

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

<u>Remarks</u>

- This is a bad method to compute the whole inverse entry by entry
- This is a good method to compute a single entry of A^{-1}.

Example (from 2/12)

Find the (3,1) entry of A^{-1} if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

$$(-1)^{3+1} C_{13} = \det \begin{bmatrix} 0 & 1 \\ 5 & 6 \end{bmatrix} = 0 - 5$$

$$= \boxed{-5}$$

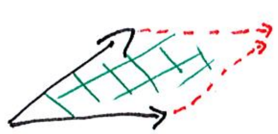
compare this to the result of the example from 2/12.

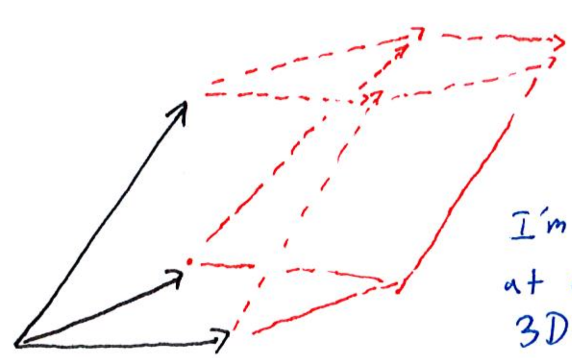
Areas and Volumes

determinants have nice geometric interpretations

Theorem

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$





I'm very bad at drawing in 3D. My bad!

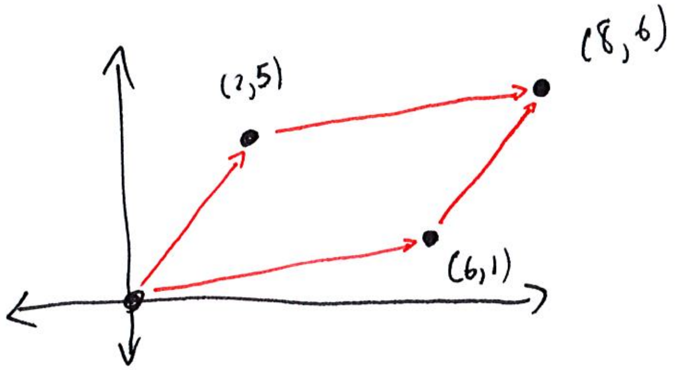
Example

Find the area of the parallelogram with vertices

$(3,3) \quad (5,8) \quad (9,4) \quad (11,9)$

want to use vectors so shift one vertex to origin. Subtract $(3,3)$ from all:

$(0,0) \quad (2,5) \quad (6,1) \quad (8,6)$



$$\left| \det \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} \right|$$

$$= |2 - 30|$$

$$= \boxed{28}$$

can use any 2 vectors (other than $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$)

$$\left| \det \begin{bmatrix} 8 & 2 \\ 6 & 5 \end{bmatrix} \right| = |40 - 12| = \boxed{28}$$

why is this?

Linear Transformations

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If $S \subseteq \mathbb{R}^n$ is a subset of the domain, let $T(S)$ denote the image of S under T

Theorem

a) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by matrix A, and $S \subset \mathbb{R}^2$, then

$$\text{area of } T(S) = |\det A| (\text{area of } S)$$

b) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by matrix A, and $S \subset \mathbb{R}^3$, then

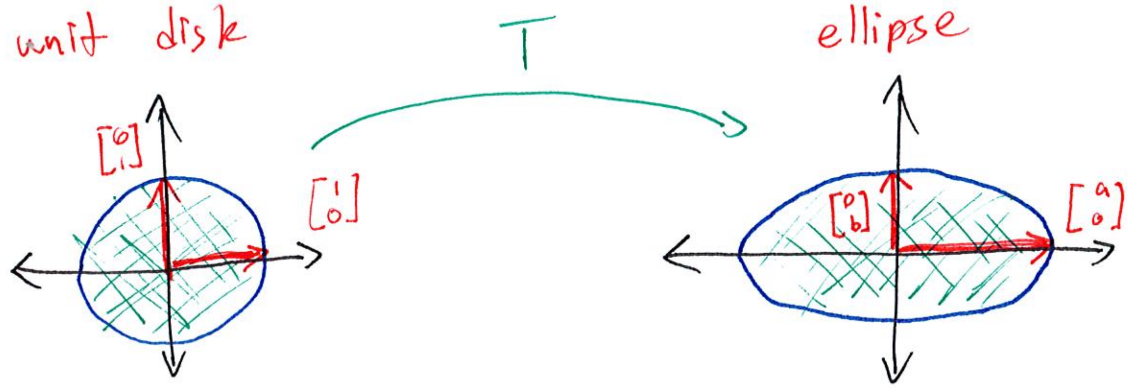
$$\text{volume of } T(S) = |\det A| (\text{volume of } S)$$

Example (area of an ellipse)

Let $a, b > 0$. What is the area of the ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \quad ?$$

Define a linear transformation:



From section §1.9 we know this transformation is given by $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Then we see

$$\text{Volume ellipse} = \left| \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right| \cdot \text{volume unit disk}$$

$$= ab \cdot \pi (1)^2$$

$$= \boxed{ab\pi}$$